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the sea as a mathematical model

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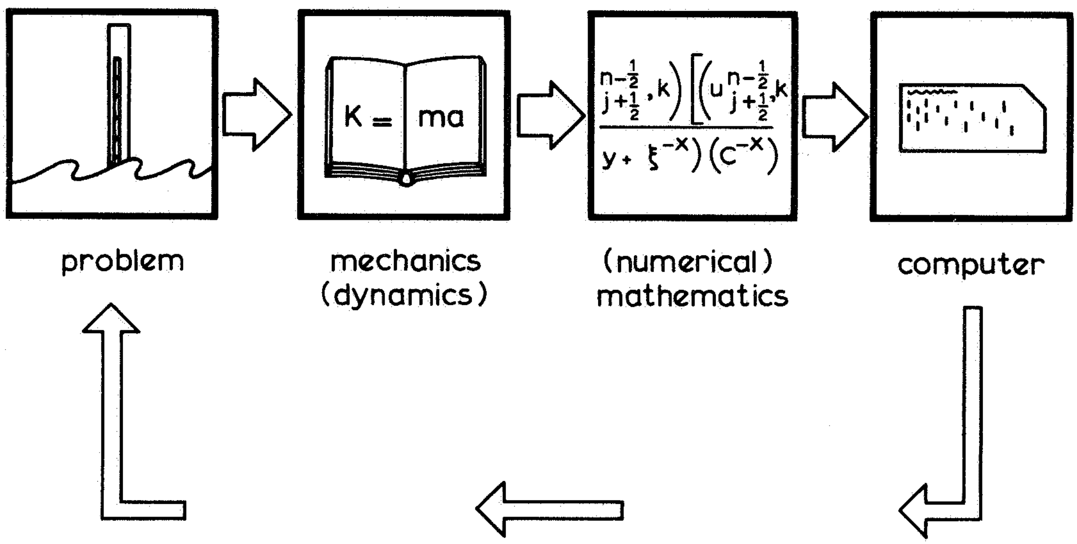
THE SEA AS A MATHEMATICAL MODEL

1 Introduction

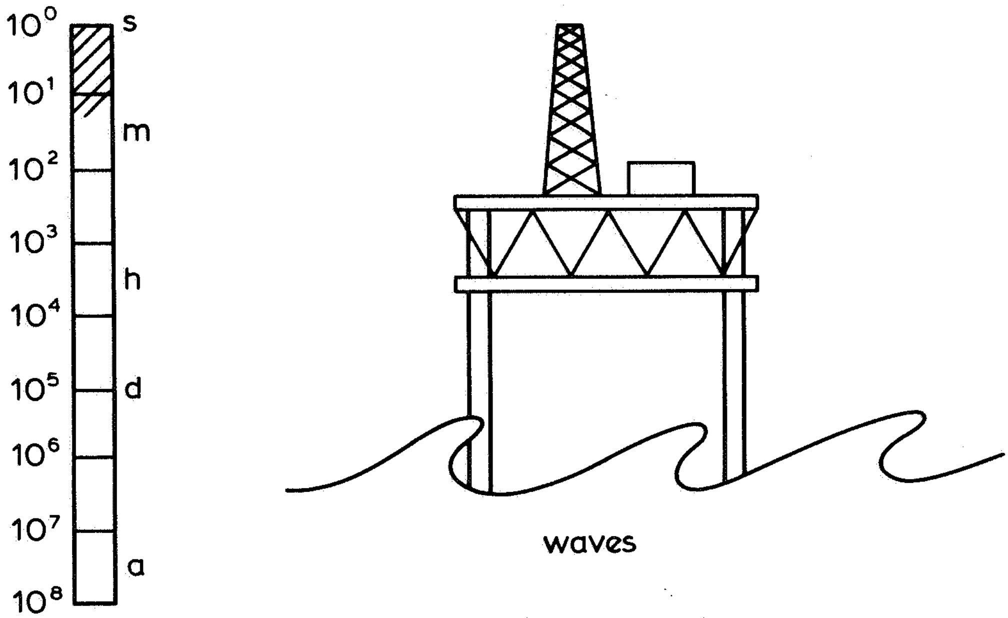
The construction of mathematical models of marine phenomena is not a recently developed technique. As early as in the course of the 18th century some of the world's greatest mathematicians, like Laplace, have tried their hand at it. It is true, however, that realistic, i.e. not too schematic, models can be prepared only since about twenty years, through the application of computers [3].

The range of computer applications is progressively widening-still, computer applicability is not unrestricted. It is still quite easy to come up with a problem which, even for the most powerful computer systems, is anything but a mere trifle. On the other hand it would not in all cases be wise to apply maximum computing power to solve problems that can be solved using simpler techniques. There should at all times be a trade-off between purpose and resources. Such trade-offs involve various disciplines, including mechanics or -in a wider sense- dynamics, and (numerical) mathematics. The representatives of the disciplines referred to above cannot carry out their work independently of eachother, no more than they are able to work independently of the people who

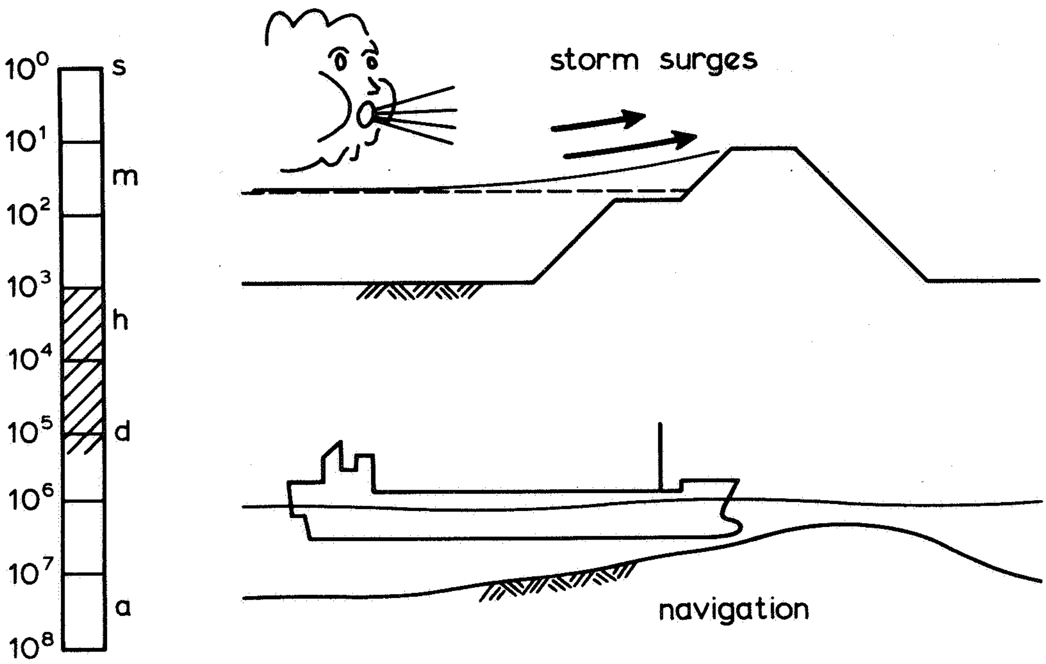


define the problem and take measurements in the field, or of the computer experts.

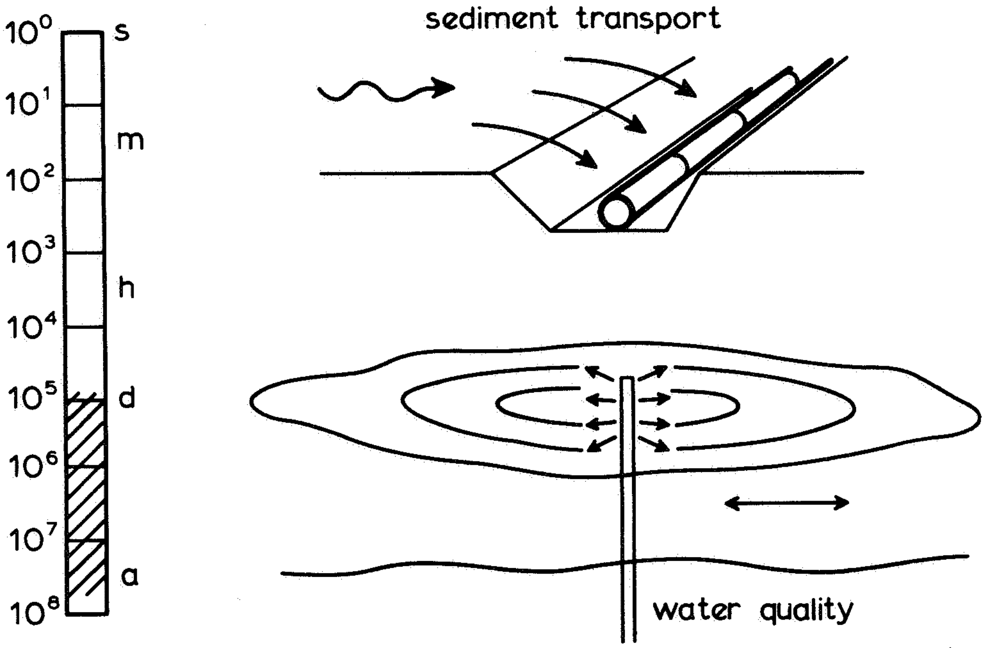
Before we discuss mathematical models of the sea we should first have some information about the purpose for which such models are needed. Well then, the problems are varying widely, relating to very short-term as well as very long-term phenomena. Some examples may be given here; I do not, however, claim that the list is exhaustive.



- With a small time scale (ranging from a few seconds to one minute) we can think of short waves, and the impact of such waves on coastal defences, offshore constructions, and navigation. Within the framework of the marine environment these phenomena are of rather indirect sifnificance, so I will not discuss them in detail; however, the following will show that waves do exert their influence on several points.



- Tides, tidal currents, and meteorological effects, such as storm surges, have a medium time scale (ranging from hours to days). Negative storm surges may also have significant effects on navigation, in view of the limited navigable depth of the North Sea.

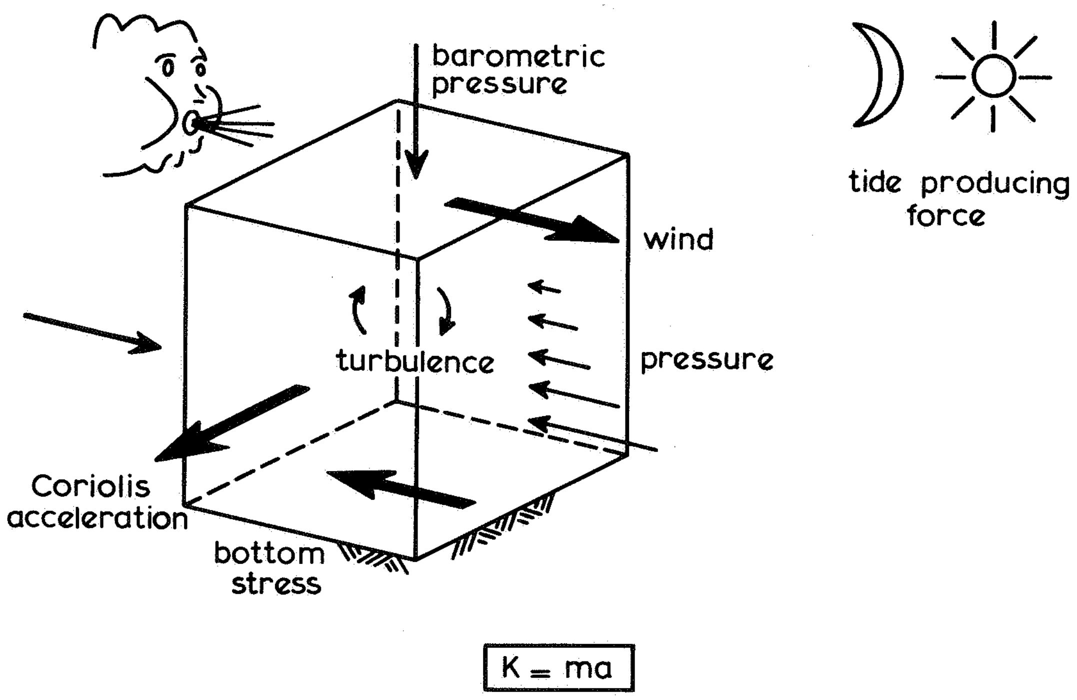


- Since the average current (residual circulation) is weak, dispersion of matter is effected over a long period of time (ranging from several months to several years). Similarly, sediment transport takes comparable lengths of time.

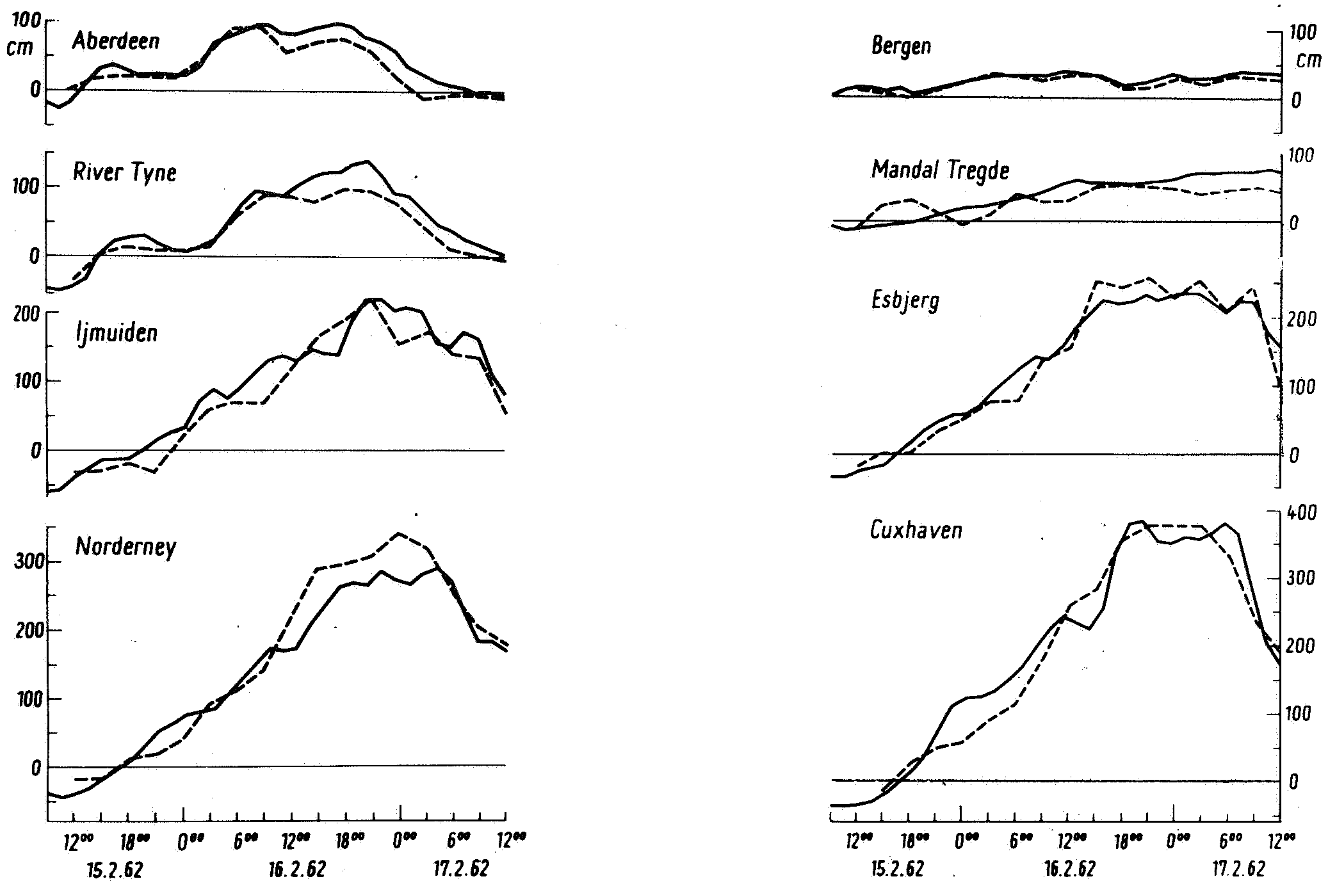
The fact that time scales are widely varying implies that there is not one single mathematical model of the sea, that may be applied to solve all problems, but that there is a whole family of models instead, each of which more or less focusses on a specific problem category.

2 Mechanics and dynamics of the marine environment

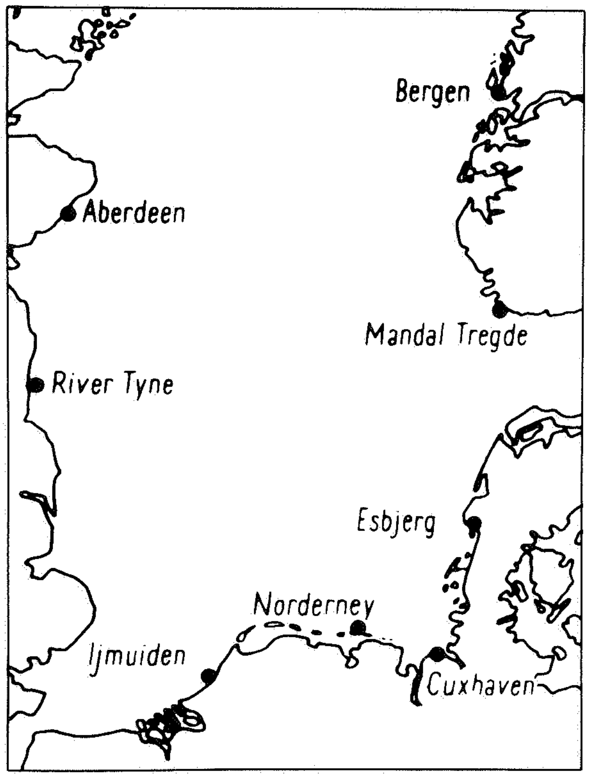
Since the motion of the marine water, besides being significant as an individual phenomenon, is also important as far as transport of other materials is involved, we will discuss the hydrodynamic aspect first. The water of the sea is subject to various forces causing water motion, as defined in Newton's law.



- Tides are caused by the gravitational attraction of sun and moon. This force is of so little significance in an area like the North Sea, that a tidal amplitude of a mere 1-2 cm would be the result [6]. Then, what causes the much stronger tide in the North Sea? It is the result of external influences entering the sea along its edges [4]. An illustrative example is given by the phenomenon of an external surge entering the North Sea near Scotland, and traveling as it were along the North Sea shoreline. Tides are propagated in a similar way.



---- observation
____ computation
((tide + wind set-up) - tide)



STORM SURGE OF 16/17 February 1962 ([4])

- Two kinds of meteorological forces can be distinguished. Differences in barometric pressure may cause circulations and differences in water level. In a state of equilibrium the water level decreases proportionately to the air pressure: the "inverted barometer" [5]. Most often the direct wind forces are much more important.

- Although it should really be classified among the accelerating forces, the ostensible force exerted by the Coriolis acceleration may be mentioned, which is perpendicular to the water speed vector, and which deflects the current in the Northern Hemisphere to the right of its motion (Buys-Ballot's Law).

- The aspect of bottom stress is especially significant in relatively shallow waters.

- Gravitation produces hydrostatic forces exerting their influence through the pressure differences.

- Internal forces, such as pressure differences due to differences in water level or in water density, and turbulent frictional forces determine the distribution of the water motion throughout the fluid layer, which makes them predominantly significant with regard to transport phenomena.

The aforementioned forces may all be expressed in mathematical formulas, resulting in a set of differential equations:

$\frac{\partial u}{\partial t} +$	inertia
$+ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} +$	convective acceleration
$- f v +$	Coriolis acceleration
$+ \frac{1}{\rho}\frac{\partial p}{\partial x} +$	pressure gradient
$- \frac{1}{\rho}\left\{\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right\} =$	internal stresses
$= 0$	

with a similar equation solving for y.

$$\frac{\partial p}{\partial z} + \rho\, g = 0$$ hydrostatic pressure

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ continuity equation

where

x,y,z = coordinates (z vertically upward, measured from the undisturbed water surface)

t = time

u,v,w = velocity components in directions x,y,z

ρ = density

p = pressure

f = Coriolis parameter

τ_{xx} etc. = turbulent stresses

g = gravitational acceleration

The above equations are commonly referred to as the Reynolds equations. These have been complemented with the shallow-water approximation, which is reflected in the hydrostatic pressure distribution, and with the Boussinesq approximation, for minor density differences. In this form, the equations do no longer permit description of short waves. The general equations of motion actually have been averaged over such a time interval, that both turbulent fluctuations and short waves have disappeared (see Section No. 6).

Differential equations are worthless if no boundary conditons are available. For the bottom and for the water level these boundary conditions are the following:

on the bottom $z = -h(x,y)$: $u = v = w = 0$

on the water surface $z = \xi(x,y,t)$:

$$p = p_a$$ barometric pressure

$$- \tau_{xx} \frac{\partial \xi}{\partial x} - \tau_{xy} \frac{\partial \xi}{\partial y} + \tau_{xz} = \tau_{sx}$$

wind shear

$$- \tau_{xy} \frac{\partial \xi}{\partial x} - \tau_{yy} \frac{\partial \xi}{\partial y} + \tau_{yz} = \tau_{sy}$$

$$\frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y} - w = 0$$ kinematic condition

In more abstract terms the equations may be considered as transport equations of horizontal momentum. Then, the parallel with the transport equation(continuity equation) for a material (or heat) becomes clear:

$\frac{\partial c}{\partial t} +$	storage
$+ u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z} +$	convective transport
$- \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial z}\right) +$	turbulent diffusion
$+ S = 0$	degradation/growth/interaction

where c = concentration (temperature). Now, this equation does not include any terms that correspond with the Coriolis acceleration and the pressure gradient. Therefore, the transport mechanism of matter, in spite of all similarity, is still quite essentially different from that of momentum. This fact is to be taken into account in mathematical models.

Water quality or ecological models consider various substances. Such transport equation applies to each of those substances. Water acts as the transporting medium for the substances, so the water quality model can be regarded as some kind of superstructure on the hydrodynamic model. There is feedback, again, to the water motion, since the density may depend on the concentrations of the substances involved, in particular if these are salt, heat, or sediments.

The water quality side contains terms (S) describing, in addition to external sources or sinks of matter, several other significant processes:

- a degradation or growth term $\pm c/t_r$ resulting in an exponential decrease (+ sign) or growth (- sign) with a time constant t_r. The latter may vary from a few hours (with some kinds of bacteria or rapidly decaying radioactivity) to many years (with less readily degradable matters);
- terms of interaction between the various substances, possibly relating to such aspects as: oxygen consumption with bio-degradation, or food chains with biological processes. Again, widely vaying time scales may play a part.

When formulating the - often non-linear - processes of growth, degradation and interaction major problems are frequently encountered. Such formulation, therefore, ranks among the principal activities in the development of water quality models. The fact that time scales of different magnitude present mathematical problems that have not been solved so far, in general.

Due to the process of averaging referred to earlier the number of unknown exceeds that of the equations, viz. the turbulent transport terms, like $\tau_{xz}, \tau_{yz}, \phi_z$. These are the principal terms, because the other terms occur as horizontal gradient, involving a scale of length that is much greater than in the vertical direction. It has been standard practice for quite some time now to describe the vertical exchange terms, on the analogy of molecular processes, using a (turbulent) viscosity ε and diffusion coefficient ε_s:

$$\tau_{xz} = \rho\, \varepsilon \frac{\partial u}{\partial z}$$

turbulence model

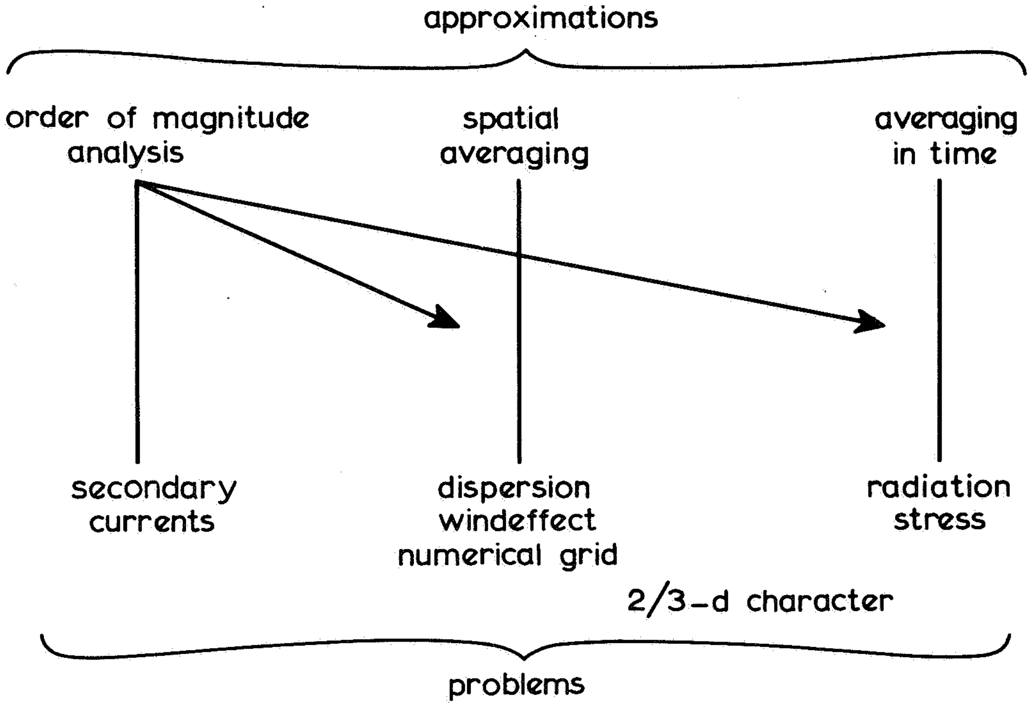
$$\phi_z = \varepsilon_s \frac{\partial c}{\partial z}$$

However, otherwise than with molecular processes, ε and ε_s are not constants. In the course of the past few years a great deal of research has been done to find either a motivation for this approach, or an alternative. The results of this research have shown that this is a good approach under certain conditions that often occur in the sea (see e.g. in [8]). The correct forms of ε and ε_s, e.g. under the influence of the wind and, consequently, waves and of possible stratification are still being investigated.

Thus, we have gradually defined a three-dimensional model of water motion and water quality. The physical basis is fairly solid; the major uncertainties are the exact behaviour of the turbulent exchange coefficients and the reacting processes of the water quality parameters. The required numerical effort is rather great; consequently, such models are still on the edge of what is practically feasible.

3 Possible approximations

What can we do to simplify the model? Actually, there are three ways to accomplish this, viz. disregarding insignificant terms; averaging over a physical area, or over time.



Evaluating the order of magnitude of the various terms of the motion and transport equations is not the easiest thing to do. By making the equations non-dimensional some dimensionless numbers may be distinguished that provide a guide to the approximate ratio between the various terms, without actually providing the real ratios because, to know the real ratios, we should know the order of magnitude as early as the moment we convert the equations into non-dimensional ones. The major components are:

Definition	Name	Order of magnitude	Measure of
$Ro = u(fh)^{-1}$	Rossby	50-500	Coriolis acceleration
$f = u(gh)^{-\frac{1}{2}}$	Froude	0.005-0.05	convective acceleration
$F_i = u(\frac{\Delta\rho}{\rho} g h)^{-\frac{1}{2}}$	Froude (internal)	0.02-1	density differences
$Ri = -\frac{g}{\rho}\frac{\partial\rho}{\partial z}(\frac{\partial u}{\partial z})^{-2}$	Richardson	0-0.5	portion of turbulent energy lost due to stratification
$Re = uh/\varepsilon$	Reynolds (turbulent)	250	turbulent friction
$Pr = \varepsilon/\varepsilon_s$	Prandtl (turbulent)	1-5	turbulent diffusion

Here, $\Delta\rho$ is a typical density difference. The numerical values have been given for an area like the North Sea. They are not so extreme as to justify the disregarding of certain terms. For instance, it will appear that the convective acceleration terms, in spite of their small order of magnitude, still play a vital part in the generation of residual circulations. It is sometimes possible, however, to examine specific phenomena from a qualitative point of view, using highly simplified models.

4 Averaging over depth

Since many seas, in particular the North Sea, are quite shallow, the method of averaging over depth is an obvious choice although the fact should not be overlooked that the vertical velocity distribution is far from uniform. As with other averaging techniques the non-linear terms now present problems. The below formula e.g. applies:

$$\overline{u\,v} = \bar{u}\,\bar{v} + \overline{(u - \bar{u})\,(v - \bar{v})}$$

where the line represents the mean. The second term represents momentum transfer, or shear stress, due to deviations from the mean velocity. A similar effect occurs with the transport equation of matter:

$$\overline{u\,c} = \bar{u}\,\bar{c} + \overline{(u - \bar{u})\,(c - \bar{c})}$$

where the second term represents an effective mass transport relative to the mean current, sometimes referred to as dispersion.

Thus averaging in the vertical direction produces the following equations for a two-dimensional model:

$\frac{\partial}{\partial t}(H\,\bar{u})\ +$	inertia
$+\ \frac{\partial}{\partial x}(H\,\bar{u}^2)\ +\ \frac{\partial}{\partial y}(H\,\bar{u}\,\bar{v})\ +$	convective acceleration
$-\ f\,H\,\bar{v}\ +$	Coriolis acceleration
$+\ \frac{1}{\rho}\,H\,\frac{\partial p_a}{\partial x} + g\,H\,\frac{\partial \xi}{\partial x} + \frac{1}{2}\,\frac{gH}{\rho}\,\frac{\partial \bar{\rho}}{\partial x} + \frac{1}{\rho}\,\frac{\partial p'}{\partial x}\ +$	pressure gradient
$-\ \frac{\tau_{sx} - \tau_{bx}}{\rho}\ +$	wind/bottom stress
$-\ \{\frac{\partial}{\partial x}(HT_{xx}) + \frac{\partial}{\partial y}(H\,T_{xy})\} = 0$	effective stresses
$\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial x}(H\,\bar{u}) + \frac{\partial}{\partial y}(H\,\bar{v}) = 0$	continuity

$$\frac{\partial}{\partial t}(H\,\bar{c}) +$$ storage

$$+ \frac{\partial}{\partial x}(H\,\bar{u}\,\bar{c}) + \frac{\partial}{\partial y}(H\,\bar{v}\,\bar{c}) +$$ convective transport

$$- \{\frac{\partial}{\partial x}(H\,F_x) + \frac{\partial}{\partial y}(H\,F_y)\} +$$ dispersion

$$+ H\,\bar{S} +$$ degradation/growth/interaction

$$- \phi_s + \phi_b = 0$$ transport through surface/bottom

where

$H = h + \xi$ = total water depth

τ_{bx}, τ_{by} = bottom shear stress

Now, the external meteorological forces have been transferred from the boundary conditions to the differential equations. The term p' is a remainder relating to the non-uniform density distribution. It becomes important with strongly stratified systems. Measurements taken in the North Sea have shown that the shallow southern part is fairly well mixed the year round. There is some stratification in the deeper northern part. This may be a reason for preparing dual-layer models for such areas.

The effective stresses and dispersive transports (through vertical planes) are:

$$T_{xy} = \bar{\tau}_{xy}/\rho - \overline{(u - \bar{u})\;(v - \bar{v})}$$

$$F_y = \bar{\phi}_y - \overline{(v - \bar{v})\;(c - \bar{c})}$$

They are often disregarded in the equations of motion, although it has lately appeared this is not permitted, for effective stresses are co-responsible for the occurence of circulation currents, such as eddy flows behind an obstacle. It is difficult to mathematically formulate these processes. More progress in this field has been made in meteorology, where similar phenomena occur, and many hydronamic models are inspired by earlier experiences in that discipline. The order of magnitude can be estimated by considering some schematic cases. An example thereof is the current under the influence of wind forces and the Coriolis acceleration [12]. The example shows that shear T_{xy} may increase to 5 per cent of the wind shear-stress, when depth is about half the so-called Ekman depth, which is approx. 150 m for the North Sea. Small as this influence may be, it may not be permissible to disregard it relative to other minor influences.

Another point is that of bottom stress, for which often the following equation is taken, on the analogy of channel flow:

$$(\tau_{bx}, \tau_{by}) = \rho \, g \, C^{-2} \, (\bar{u}^2 + \bar{v}^2)^{\frac{1}{2}} \, (\bar{u} \, , \, \bar{v})$$

(C = Chézy coefficient of bottom roughness).
The above is true only if the depth, relative to the Ekman depth is small, and if there is no wind influence. In other cases both the direction and the magnitude of the bottom stress may deviate strongly. The effect of wind on channel waters has been theoretically investigated [10]. The theory may be extended to the two-dimensional situation, with a result that fairly corresponds with the expression (mentioned in [2]):

$$(\tau_{bx}, \tau_{by}) = - \, m \, (\tau_{sx}, \tau_{sy}) + s \, \rho \, (\bar{u}^2 + \bar{v}^2)^{\frac{1}{2}} \, (\bar{u} \, , \, \bar{v})$$

Some data is available on the dispersion terms F_x and F_y in simple situations, such as flow through a pipe, or flow through a straight channel [11]. The available information shows that they again act in a diffusion-like way:

$$F_x = D \frac{\partial \bar{c}}{\partial x}$$

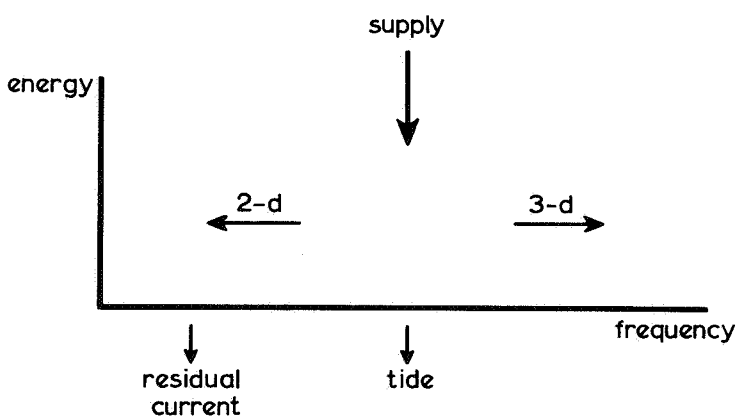
So far, it is not quite clear what technique should be applied to generalize these terms for two-dimensional tidal currents as these occur in the sea.

5 Averaging in the numerical process

With almost any numerical technique the area is subdivided into small sections having the dimensions Δx, Δy, to which averaging is in some or other way applied, in actual fact. Much the same way as with averaging over depth effective shears and transports are again found, that this time depend on the mesh width. Although these effects proper need not be great, their being disregarded results in the following undesired phenomena.

Due to the non-linear behaviour of the water motion harmonics are generated with periodic motions. More generally, large-scale motions cause secondary motions on a smaller scale, i.e. energy is actually transferred from larger to smaller scales. This process continues until scales are so small that energy in the mechanical form disappears by conversion into heat. Usually, it is not possible by far to represent these smallest scales in a numerical grid. The transfer of energy, which also occurs in the mathematical model, is therefore interrupted at the smallest scale that can still be represented numerically, which is twice the mesh width. This may become manifest as explosive growth of small waves or eddies, which phenomenon is commonly referred to as non-linear instability. To produce a properly working model, therefore, the transfer of energy to smaller scales is to be simulated, and for that purpose we need the effective shears referred to earlier [1].

Research into these phenomena is continuing. One complication remains in the fact, that the current in a relatively very shallow sea, like the North Sea, could very well be behaving like a two-dimensional turbulence, showing- qualitatively speaking- a behavioural pattern different from that of three-dimensional turbulence we were just discussing. Here, the part of energy transfer is taken over by a transport of vorticity, whereas energy is rather transferred to larger-scale motions.



The effective mass transports due to application of the mean value theorem to the numerical grid depend, of course, on the mesh width. Since they are the result of averaging the convective processes, it is obvious that they decrease in importance as the water motion proper is better represented in the model, which will be the case when the mesh width is proportionately smaller.

6 Averaging over time

Tidal current is not directly important to longer-term applications, such as water quality studies. The obvious thing to do, then, is to take an average over such a long period of time, that only the residual circulation and the mean water level remain. In fact, something similar has been done for the short waves in the three-dimensional model. It will be obvious now, that this will again result in "remainders" within the equations, as with spatial averaging. With short waves these remainders are known as "radiation stresses" [7]; with longer waves, such as tides, they are called "tidal stresses" [9], although in both cases the same phenomena are meant. Residual circulations and mean water levels are still variable, but at such low rates that a quasi-steady flow may be assumed.

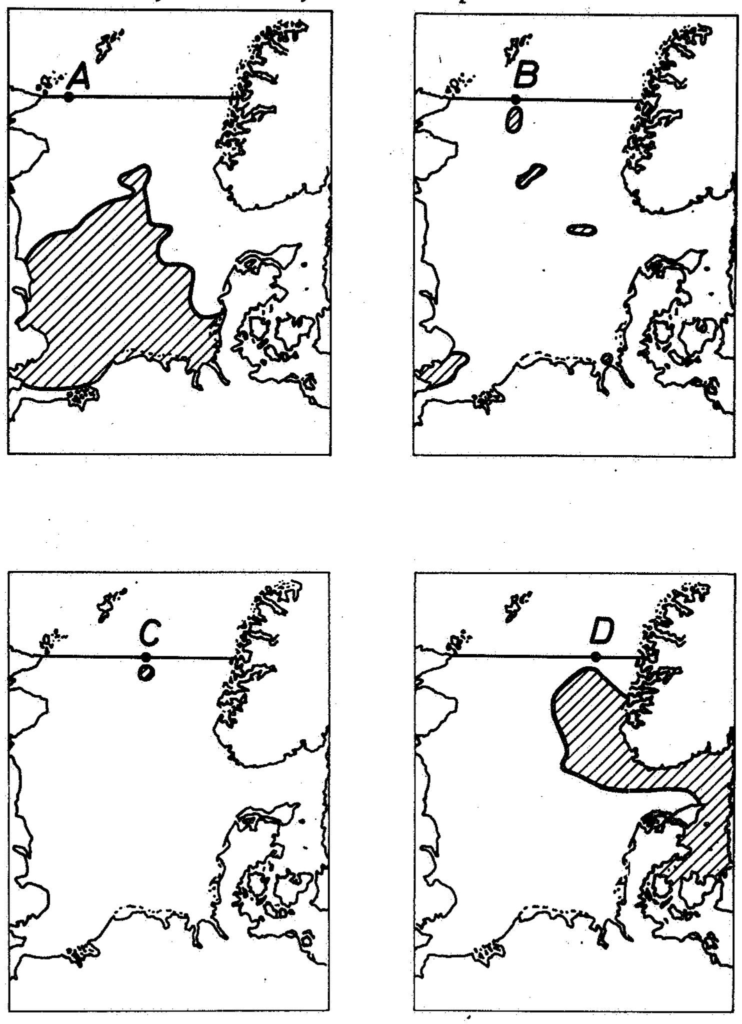
The equations for a two-dimensional model, with a time mean value, are not included here; they are analogous to the preceeding ones. Further details may be found in [12]. In any case tidal stresses play a significant part, which again shows that the convective terms, in spite of the fact that they are not so large in themselves, are essential. Another major factor, at least in shallow areas, is bottom friction. This friction is due to an oscillating (tidal) motion with a small residual velocity component. This value may be fairly well approximated, by assuming it is proportionate te the residual velocity (in terms of magnitude and direction) and to the amplitude of the oscillating velocity component. In practice, tidal stresses and bottom friction are computed from a numerical tide computation (two-dimensional, time-dependent model) which, depending on the desired accuracy, extends either over one tidal period, or over one full cycle between springtide and neap tide. Subsequently the residual circulation equations may be solved [9].

In the model with time averages, the mass transport terms are very important. This has been the subject of quite some empirical and theoretical research. It would exceed the scope of this paper to discuss this matter in detail. One alternative is the application of a similar technique to two time scales, as described above.

7 Boundary conditions, and data

The fact that boundary conditions are important was discussed aearlier in this paper. With the varying models it is not always equally simple to define the boundary conditions in such a way that what is called a mathematically well posed problem is produced. Another point still is, the way these boundary conditions should be handled numerically.

To the user the principal question is- to what degree of accuracy must boundary conditions be supplied; how sensitive are the results of the model to those conditions, and to what extent is the model influenced by those conditions. It is not easy to provide theorical answers to these questions. Often, the sole possibility is, to carry out numerical experiments. Thus, influence functions have been determined for the tide on the northern edge of the North Sea [4]; there are, however, other aspects on which few data is available.

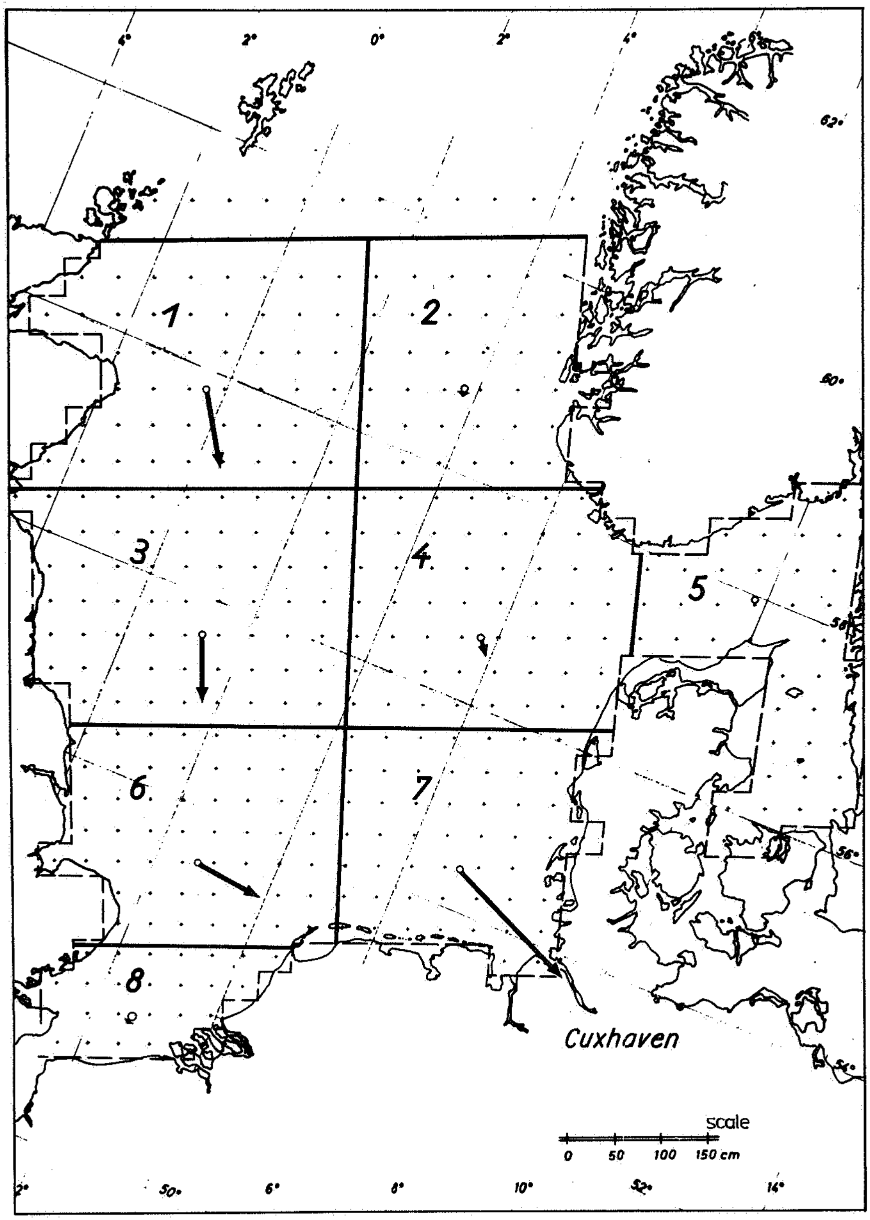


Lines connecting points of equal water level increases of 0.1 m Cross-hatched areas with disturbances in excess of 0.1 m, 17 hours 20 minutes after start of disturbance. Points A,B,C,D indicate sources of disturbances.

Some examples are e.g. the method of representation of the coast line and the shallow coastal strip in numerical grids. The effect of inaccuracies on the edge may spread over rather large distances. Therefore, it is vitally important to have information about the size of such inaccuracies. Similar questions may be raised with regard to other data that is required for a model, such as

- bottom schematication: accuracy, mesh width, bottom roughness;
- wind field: direction, size, variations in time and position.

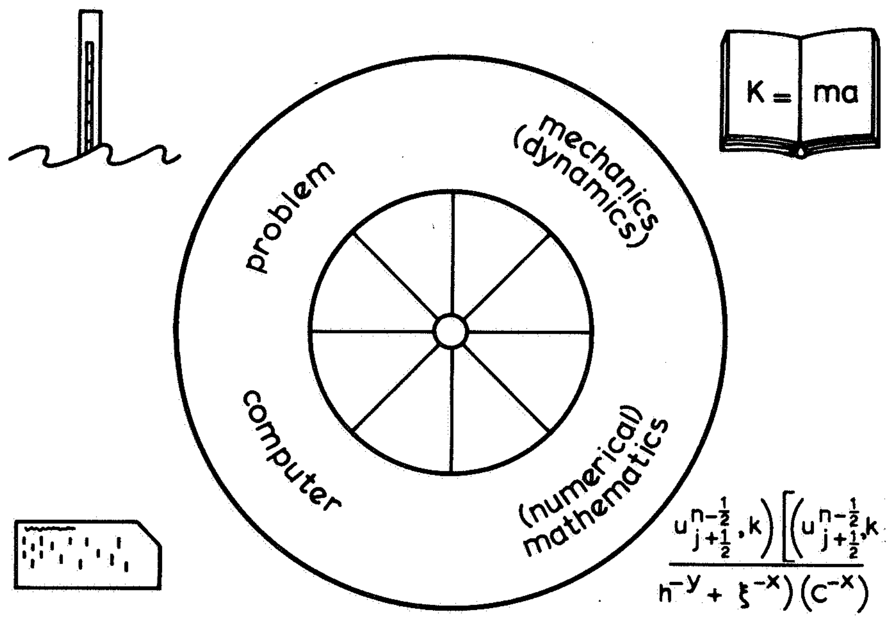
Such matters do, of course have an effect on the water quality part of the models. However, very few data is available on these effects.



The arrows point in the direction of the wind flow causing the largest set up at Cuxhaven. The length of the arrows is proportionate with the water level increase thus produced at Cuxhaven. Wind speed is constant in all areas.

8 Conclusion

The various topics dealt with in this paper will have made it quite clear that mathematical models of marine phenomena are anything but matters that can be set up and handled by several experts working independently. There is, on the contrary, a close relationship between the definition of the problem, the physical-mathematical formulation thereof, the numerical technique, the data and the utilization of models, let alone calibrating, verifying and interpreting the results.



Consequently, such mathematical models will have to be designed and used in close cooperation by the definer of the problem, the physicist and the mathematician. Inasfar as these three disciplines are not the combined specialism of one single individual, these experts must have sufficient knowledge of the discipline the others are working in to permit them to achieve meaningful results.

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